

The Wehrl entropy has Gaussian optimizers

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Abstract: We determine the minimum Wehrl entropy among the quantum states with a given von Neumann entropy, and prove that it is achieved by thermal Gaussian states. This result determines the relation between the von Neumann and the Wehrl entropies. The key idea is proving that the quantum-classical channel that associates to a quantum state its Husimi Q representation is asymptotically equivalent to the Gaussian quantum-limited amplifier with infinite amplification parameter. This equivalence also permits to determine the $p \rightarrow q$ norms of the aforementioned quantum-classical channel, and prove that they are achieved by thermal Gaussian states. The same equivalence permits to prove that the Husimi Q representation of a passive state (i.e. a state diagonal in the Fock basis with eigenvalues decreasing as the energy increases) majorizes the Husimi Q representation of any other state with the same spectrum, i.e. it maximizes any convex functional.

1. Introduction

The Husimi Q representation [1] is a probability distribution in phase space that describes a quantum state of a Gaussian quantum system, such as an harmonic oscillator or a mode of electromagnetic radiation [2, 3]. It coincides with the probability distribution of the outcomes of a heterodyne measurement [4] performed on the state. This measurement is fundamental in the field of quantum optics. It is used for quantum tomography [5], and it lies at the basis of an easily realizable quantum key distribution scheme [6, 7]. The Husimi Q representation is also used to study quantum effects in superconductors [8].

The Wehrl entropy [9, 10] of a quantum state is the Shannon differential entropy [11] of its Husimi Q representation. It is considered as the classical entropy of the state, and it coincides with the Shannon differential entropy of the outcome of a heterodyne measurement performed on the state.

While the von Neumann entropy of any pure state is zero, this is not the case for the Wehrl entropy. E. Lieb proved [12, 13] that the Wehrl entropy is minimized by the Glauber coherent states [14–18], with a proof based on some difficult theorems in Fourier analysis. This result has then been generalized to symmetric $SU(N)$ coherent states [19, 20]. It has also been proven [20–22] that the Husimi Q representation of coherent states majorizes the Husimi Q representation of any other quantum state, i.e. it maximizes any convex functional.

Let us now suppose to fix the von Neumann entropy of the quantum state. What is its minimum possible Wehrl entropy? We determine it and prove that it is achieved by the thermal Gaussian state with the given von Neumann entropy (Theorem 7). This result determines the relation between the von Neumann and the Wehrl entropies. The key idea is proving that the quantum-classical channel that associates to a quantum state its Husimi Q representation is asymptotically equivalent to the Gaussian quantum-limited amplifier with infinite amplification parameter (Theorem 4). We can then link this equivalence to the recent results on quantum Gaussian channels [23–26], noncommutative generalizations of the theorems in Fourier analysis used in Lieb’s original proof. This link also permits to determine the $p \rightarrow q$ norms of the aforementioned quantum-classical channel, and prove that they are finite for $1 \leq p \leq q$, infinite for $p > q \geq 1$, and in any case achieved by thermal Gaussian states (Theorem 6). Moreover, the same link implies that the Husimi Q representation of a passive state (i.e. a state diagonal in the Fock basis with the eigenvalues decreasing as the energy increases) majorizes the Husimi Q representation of any other state with the same spectrum, i.e. it maximizes any convex functional (Theorem 5).

The paper is structured as follows. Section 2 introduces Gaussian quantum systems and Gaussian quantum states, and Section 3 introduces the Husimi Q representation. Section 4 defines the Gaussian quantum-limited amplifier and presents the recent results on quantum Gaussian channels needed for the proofs. Section 5 presents our results, that are proved in Sections 6, 7, 8 and 9. Section 10 draws the conclusions. Appendix A defines the Gaussian quantum-limited attenuator, that is needed for some of the proofs. Appendix B contains some auxiliary theorems and lemmas.

2. Gaussian quantum systems

We consider the Hilbert space of M harmonic oscillators, or M modes of the electromagnetic radiation, i.e. the irreducible representation of the canonical commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \mathbb{I}, \quad i, j = 1, \dots, M. \quad (1)$$

The operators $\hat{a}_1^\dagger \hat{a}_1, \dots, \hat{a}_M^\dagger \hat{a}_M$ have integer spectrum and commute. Their joint eigenbasis is the Fock basis $\{|n_1 \dots n_M\rangle\}_{n_1, \dots, n_M \in \mathbb{N}}$. The Hamiltonian

$$\hat{N} = \sum_{i=1}^M \hat{a}_i^\dagger \hat{a}_i \quad (2)$$

counts the number of excitations, or photons.

One-mode thermal Gaussian states have density matrix

$$\hat{\omega}_z = \sum_{n=0}^{\infty} (1-z) z^n |n\rangle\langle n|, \quad 0 \leq z < 1. \quad (3)$$

Their average energy is

$$E = \text{Tr} [\hat{N} \hat{\omega}_z] = \frac{z}{1-z}, \quad (4)$$

and their von Neumann entropy is

$$S = -\text{Tr} [\hat{\omega}_z \ln \hat{\omega}_z] = (E+1) \ln(E+1) - E \ln E := g(E). \quad (5)$$

3. The Husimi Q representation and Wehrl entropy

The classical phase space associated to a M -mode Gaussian quantum system is \mathbb{C}^M , and for any $\mathbf{z} \in \mathbb{C}^M$ we define the coherent state

$$|\mathbf{z}\rangle = e^{-\frac{|\mathbf{z}|^2}{2}} \sum_{n_1, \dots, n_M \in \mathbb{N}} \frac{z_1^{n_1} \dots z_M^{n_M}}{\sqrt{n_1! \dots n_M!}} |n_1 \dots n_M\rangle. \quad (6)$$

Coherent states are not orthogonal:

$$\langle \mathbf{z} | \mathbf{w} \rangle = e^{\mathbf{z}^\dagger \mathbf{w} - \frac{|\mathbf{z}|^2 + |\mathbf{w}|^2}{2}} \quad \forall \mathbf{z}, \mathbf{w} \in \mathbb{C}^M, \quad (7)$$

but they are complete and satisfy the resolution of the identity [22]

$$\int_{\mathbb{C}^M} |\mathbf{z}\rangle\langle \mathbf{z}| \frac{d^{2M}z}{\pi^M} = \hat{\mathbb{I}}, \quad (8)$$

where the integral converges in the weak topology. The POVM associated with the resolution of the identity (8) is called heterodyne measurement [4].

Definition 1 (Husimi Q representation). *The Husimi Q representation of a quantum state $\hat{\rho}$ is the probability distribution on phase space of the outcome of an heterodyne measurement performed on $\hat{\rho}$, with density*

$$Q(\hat{\rho})(\mathbf{z}) := \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle, \quad \mathbf{z} \in \mathbb{C}^M, \quad \int_{\mathbb{C}^M} Q(\hat{\rho})(\mathbf{z}) \frac{d^{2M}z}{\pi^M} = 1. \quad (9)$$

Definition 2 (Wehrl entropy). *The Wehrl entropy of a quantum state $\hat{\rho}$ is the Shannon differential entropy of its Husimi Q representation*

$$S(Q(\hat{\rho})) = - \int_{\mathbb{C}^M} Q(\hat{\rho})(\mathbf{z}) \ln Q(\hat{\rho})(\mathbf{z}) \frac{d^{2M}z}{\pi^M}. \quad (10)$$

4. The Gaussian quantum-limited amplifier

The M -mode Gaussian quantum-limited amplifier $\mathcal{A}_\kappa^{\otimes M}$ with amplification parameter $\kappa \geq 1$ performs a two-mode squeezing on the input state $\hat{\rho}$ and the vacuum state of a M -mode ancillary Gaussian system B with ladder operators $\hat{b}_1, \dots, \hat{b}_M$:

$$\mathcal{A}_\kappa^{\otimes M}(\hat{\rho}) = \text{Tr}_B \left[\hat{U}_\kappa (\hat{\rho} \otimes |\mathbf{0}\rangle\langle\mathbf{0}|) \hat{U}_\kappa^\dagger \right]. \quad (11)$$

The squeezing unitary operator

$$\hat{U}_\kappa = \exp \left(\text{arccosh} \sqrt{\kappa} \sum_{i=1}^M \left(\hat{a}_i^\dagger \hat{b}_i^\dagger - \hat{a}_i \hat{b}_i \right) \right) \quad (12)$$

acts on the ladder operators as

$$\hat{U}_\kappa^\dagger \hat{a}_i \hat{U}_\kappa = \sqrt{\kappa} \hat{a}_i + \sqrt{\kappa - 1} \hat{b}_i^\dagger, \quad (13)$$

$$\hat{U}_\kappa^\dagger \hat{b}_i \hat{U}_\kappa = \sqrt{\kappa - 1} \hat{a}_i^\dagger + \sqrt{\kappa} \hat{b}_i, \quad i = 1, \dots, M. \quad (14)$$

The Gaussian quantum-limited amplifier preserves the set of thermal Gaussian states, i.e. for any $0 \leq z < 1$

$$\mathcal{A}_\kappa(\hat{\omega}_z) = \hat{\omega}_{z'}, \quad z' = 1 - \frac{1 - z}{\kappa}. \quad (15)$$

We now recall the latest results on quantum Gaussian channels, on which the proofs of Theorems 5, 6 and 7 are based. The proof of Theorem 5 is based on

Definition 3 (Passive rearrangement [23]). *The passive rearrangement of the quantum state*

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n |\psi_n\rangle\langle\psi_n|, \quad p_0 \geq p_1 \geq \dots \geq 0, \quad \langle\psi_m|\psi_n\rangle = \delta_{mn} \quad (16)$$

of a one-mode Gaussian quantum system is the state with the same spectrum with minimum average energy, i.e.

$$\hat{\rho}^\downarrow := \sum_{n=0}^{\infty} p_n |n\rangle\langle n|. \quad (17)$$

We say that $\hat{\rho}$ is passive if $\hat{\rho} = \hat{\rho}^\downarrow$, i.e. if $\hat{\rho}$ is diagonal in the Fock basis with eigenvalues decreasing as the energy increases.

Theorem 1 ([23]). *For $M = 1$, the output generated by a passive input state majorizes the output generated by any other input state with the same spectrum, i.e. for any quantum state $\hat{\rho}$ and any convex function $f : [0, 1] \rightarrow \mathbb{R}$ with $f(0) = 0$*

$$\text{Tr } f(\mathcal{A}_\kappa(\hat{\rho})) \leq \text{Tr } f(\mathcal{A}_\kappa(\hat{\rho}^\downarrow)). \quad (18)$$

Remark 1. Theorem 1 does not hold for $M > 1$ [27].

The proofs of Theorems 6 and 7 are based on the following conjectures, that have been proven for $M = 1$ in Refs. [25, 26].

Definition 4 (Schatten norm [28, 29]). For any $p \geq 1$ the Schatten p norm of the positive semidefinite operator \hat{A} is

$$\|\hat{A}\|_p := \left(\text{Tr } \hat{A}^p \right)^{\frac{1}{p}}. \quad (19)$$

Conjecture 2 For any $\kappa \geq 1$ and any $p, q \geq 1$ the $p \rightarrow q$ norm of $\mathcal{A}_\kappa^{\otimes M}$ is achieved by thermal Gaussian states, i.e. for any quantum state $\hat{\rho}$

$$\frac{\|\mathcal{A}_\kappa^{\otimes M}(\hat{\rho})\|_q}{\|\hat{\rho}\|_p} \leq \left(\sup_{0 \leq z < 1} \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} \right)^M. \quad (20)$$

Remark 2. Conjecture 2 has been proven in Ref. [25] for $M = 1$, and in Refs. [21, 22] for $p = 1$ and any M . In the latter case it has also been shown that the supremum in (20) is achieved by the vacuum state, i.e. in $z = 0$.

Conjecture 3 Thermal Gaussian states minimize the output von Neumann entropy of the Gaussian quantum-limited amplifier among all the input states with a given entropy, i.e. for any quantum state $\hat{\rho}$

$$S(\mathcal{A}_\kappa^{\otimes M}(\hat{\rho})) \geq M g \left(\kappa g^{-1} \left(\frac{S(\hat{\rho})}{M} \right) + \kappa - 1 \right), \quad (21)$$

with g as in (5), and where $S(\hat{\rho}) = -\text{Tr}[\hat{\rho} \ln \hat{\rho}]$.

Remark 3. Conjecture 3 has first been formulated in Ref. [30]. It has been proven for $M = 1$ in Ref. [26], and for $S(\hat{\rho}) = 0$ and any M in Refs. [22, 31]. A non-optimal lower bound to the output entropy of the Gaussian quantum-limited amplifier valid for any M is provided by the quantum Entropy Power Inequality of Refs. [30, 32].

5. Main results

We start with the asymptotic equivalence between the Husimi Q representation and the Gaussian quantum-limited amplifier, the key idea of the proofs of the other results.

Theorem 4 (Husimi-amplifier equivalence). The quantum-classical channel that associates to a quantum state its Husimi Q representation is asymptotically equivalent to the Gaussian quantum-limited amplifier with infinite amplification parameter, i.e. for any quantum state $\hat{\rho}$ and any real convex function $f \in C^1([0, 1])$ with $f(0) = 0$

$$\int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} = \lim_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}))}{\kappa^M}. \quad (22)$$

Proof. See Section 6.

Remark 4. Since for any quantum state $\hat{\rho}$

$$0 \leq \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle \leq 1, \quad (23)$$

and from Lemma 5

$$0 \leq \kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}) \leq \hat{\mathbb{I}}, \quad (24)$$

the function f in Theorem 4 needs to be defined in $[0, 1]$ only.

Theorem 5 (majorization). *For $M = 1$, the Husimi Q representation of a passive state majorizes the Husimi Q representation of any other state with the same spectrum, i.e. for any quantum state $\hat{\rho}$ and any real convex function $f \in C^1([0, 1])$ with $f(0) = 0$*

$$\int_{\mathbb{C}} f(\langle z | \hat{\rho} | z \rangle) \frac{d^2 z}{\pi} \leq \int_{\mathbb{C}} f(\langle z | \hat{\rho}^\downarrow | z \rangle) \frac{d^2 z}{\pi}. \quad (25)$$

Proof. See Section 7.

Remark 5. We state Theorem 5 for $M = 1$ only since its proof relies on Theorem 1, that does not hold for $M > 1$.

Theorem 6 ($p \rightarrow q$ norms). *Assuming Conjecture 2, for any $1 \leq p \leq q$ the $p \rightarrow q$ norm of the quantum-classical channel that associates to a quantum state its Husimi Q representation is achieved by thermal Gaussian states, i.e. for any M -mode quantum state $\hat{\rho}$*

$$\frac{\|Q(\hat{\rho})\|_q}{\|\hat{\rho}\|_p} \leq \left(\sup_{0 \leq z < 1} \frac{\|Q(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} \right)^M = \left(\sup_{0 \leq z < 1} \frac{(1 - z^p)^{\frac{1}{p}}}{q^{\frac{1}{q}} (1 - z)^{\frac{1}{q}}} \right)^M, \quad (26)$$

where

$$\|Q(\hat{\rho})\|_q = \left(\int_{\mathbb{C}^M} Q(\hat{\rho})(\mathbf{z})^q \frac{d^{2M} z}{\pi^M} \right)^{\frac{1}{q}} \quad (27)$$

is the norm of $Q(\hat{\rho})$ in $L^q(\mathbb{C}^M)$. The supremum in (26) and hence the $p \rightarrow q$ norm are finite if $1 \leq p \leq q$, and infinite if $p > q \geq 1$.

Proof. See Section 8.

Remark 6. We recall that Conjecture 2 has been proven both for $M = 1$ and for $p = 1$ and any M .

Finally, here is the main result of this paper, that determines the relation between the von Neumann and the Wehrl entropies.

Theorem 7 (Wehrl entropy has Gaussian optimizers). *Assuming Conjecture 3, the minimum Wehrl entropy among all the quantum states with a given von Neumann entropy is achieved by thermal Gaussian states, i.e. for any quantum state $\hat{\rho}$*

$$S(Q(\hat{\rho})) \geq M \left(\ln \left(g^{-1} \left(\frac{S(\hat{\rho})}{M} \right) + 1 \right) + 1 \right), \quad (28)$$

with g defined in (5).

Proof. See Section 9.

Remark 7. We recall that Conjecture 3 has been proven both for $M = 1$ and for $S(\hat{\rho}) = 0$ and any M .

6. Proof of Theorem 4

Let us first prove that

$$\int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} \leq \liminf_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}))}{\kappa^M}. \quad (29)$$

The proof is based on the following:

Theorem 8 (Berezin-Lieb inequality [33]). *For any trace-class operator $0 \leq \hat{A} \leq \hat{\mathbb{I}}$ and any convex function $f : [0, 1] \rightarrow \mathbb{R}$*

$$\int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{A} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} \leq \text{Tr } f(\hat{A}). \quad (30)$$

Proof. Let us diagonalize \hat{A} :

$$\hat{A} = \sum_{n \in \mathbb{N}} a_n |\psi_n\rangle\langle\psi_n|, \quad 0 \leq a_n \leq 1, \quad \langle\psi_m|\psi_n\rangle = \delta_{mn}, \quad \sum_{n \in \mathbb{N}} |\psi_n\rangle\langle\psi_n| = \hat{\mathbb{I}}. \quad (31)$$

We then have

$$f(\langle \mathbf{z} | \hat{A} | \mathbf{z} \rangle) = f\left(\sum_{n \in \mathbb{N}} |\langle \mathbf{z} | \psi_n \rangle|^2 a_n\right) \leq \sum_{n \in \mathbb{N}} |\langle \mathbf{z} | \psi_n \rangle|^2 f(a_n), \quad (32)$$

where we have applied Jensen's inequality to f and we have noticed that the completeness relation for the set $\{|\psi_n\rangle\}_{n \in \mathbb{N}}$ implies for any $\mathbf{z} \in \mathbb{C}^M$

$$\sum_{n \in \mathbb{N}} |\langle \mathbf{z} | \psi_n \rangle|^2 = \langle \mathbf{z} | \mathbf{z} \rangle = 1. \quad (33)$$

It follows that

$$\int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{A} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} \leq \sum_{n \in \mathbb{N}} \int_{\mathbb{C}^M} |\langle \mathbf{z} | \psi_n \rangle|^2 f(a_n) \frac{d^{2M}z}{\pi^M} = \sum_{n \in \mathbb{N}} f(a_n) = \text{Tr } f(\hat{A}), \quad (34)$$

where we have used that for the completeness relation (8) for any $n \in \mathbb{N}$

$$\int_{\mathbb{C}^M} |\langle \mathbf{z} | \psi_n \rangle|^2 \frac{d^{2M}z}{\pi^M} = \langle \psi_n | \psi_n \rangle = 1. \quad (35)$$

From Lemma 5 we have $0 \leq \kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}) \leq \hat{\mathbb{I}}$. We can then apply Theorem 8 to $\hat{A} = \kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho})$ and get

$$\begin{aligned} \text{Tr } f(\kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho})) &\geq \int_{\mathbb{C}^M} f(\kappa^M \langle \mathbf{z} | \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}) | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} \\ &= \int_{\mathbb{C}^M} f(\langle \mathbf{z} / \sqrt{\kappa} | \hat{\rho} | \mathbf{z} / \sqrt{\kappa} \rangle) \frac{d^{2M}z}{\pi^M} \\ &= \kappa^M \int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M}, \end{aligned} \quad (36)$$

where we have used Lemma 4. The claim (29) then follows taking the limit $\kappa \rightarrow \infty$.

Let us now prove that

$$\int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} \geq \limsup_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}))}{\kappa^M}. \quad (37)$$

The proof follows from the following:

Theorem 9 (Berezin-Lieb inequality [33]). *For any convex function $f : [0, 1] \rightarrow \mathbb{R}$ with $f(0) = 0$ and any integrable function $\phi : \mathbb{C}^M \rightarrow [0, 1]$*

$$\text{Tr } f\left(\int_{\mathbb{C}^M} \phi(\mathbf{z}) |\mathbf{z}\rangle\langle\mathbf{z}| \frac{d^{2M}z}{\pi^M}\right) \leq \int_{\mathbb{C}^M} f(\phi(\mathbf{z})) \frac{d^{2M}z}{\pi^M}. \quad (38)$$

Proof. See e.g. [21], Appendix B.

Let us define the measure-reprepare channel

$$\begin{aligned} \mathcal{M}_\kappa^{\otimes M}(\hat{\rho}) &= \int_{\mathbb{C}^M} \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle |\sqrt{\kappa} \mathbf{z}\rangle\langle\sqrt{\kappa} \mathbf{z}| \frac{d^{2M}z}{\pi^M} \\ &= \int_{\mathbb{C}^M} \frac{\langle \mathbf{z} / \sqrt{\kappa} | \hat{\rho} | \mathbf{z} / \sqrt{\kappa} \rangle}{\kappa^M} |\mathbf{z}\rangle\langle\mathbf{z}| \frac{d^{2M}z}{\pi^M}. \end{aligned} \quad (39)$$

We can apply Theorem 9 to

$$\phi(\mathbf{z}) = \langle \mathbf{z} / \sqrt{\kappa} | \hat{\rho} | \mathbf{z} / \sqrt{\kappa} \rangle \leq 1, \quad \mathbf{z} \in \mathbb{C}^M. \quad (40)$$

We have

$$\text{Tr } f(\kappa^M \mathcal{M}_\kappa^{\otimes M}(\hat{\rho})) \leq \int_{\mathbb{C}^M} f(\langle \mathbf{z} / \sqrt{\kappa} | \hat{\rho} | \mathbf{z} / \sqrt{\kappa} \rangle) \frac{d^{2M}z}{\pi^M}, \quad (41)$$

hence

$$\int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} \geq \limsup_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa^M \mathcal{M}_\kappa^{\otimes M}(\hat{\rho}))}{\kappa^M}. \quad (42)$$

We define for any $\mathbf{z} \in \mathbb{C}^M$ the displacement operator [3]

$$\hat{D}(\mathbf{z}) := \exp\left(\sum_{i=1}^M (z_i \hat{a}_i^\dagger - z_i^* \hat{a}_i)\right), \quad (43)$$

that acts on coherent states as

$$\hat{D}(\mathbf{z}) |\mathbf{w}\rangle = e^{\frac{\mathbf{w}^\dagger \mathbf{z} - \mathbf{z}^\dagger \mathbf{w}}{2}} |\mathbf{z} + \mathbf{w}\rangle, \quad \mathbf{z}, \mathbf{w} \in \mathbb{C}^M. \quad (44)$$

Let us define the channel

$$\begin{aligned} \mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) &= \int_{\mathbb{C}^M} \kappa^M e^{-\kappa |\mathbf{z}|^2} \hat{D}(\mathbf{z}) \hat{\rho} \hat{D}(\mathbf{z})^\dagger \frac{d^{2M}z}{\pi^M} \\ &= \int_{\mathbb{C}^M} e^{-|\mathbf{z}|^2} \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right) \hat{\rho} \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right)^\dagger \frac{d^{2M}z}{\pi^M}. \end{aligned} \quad (45)$$

Lemma 1. For any $\kappa \geq 1$

$$\mathcal{M}_\kappa^{\otimes M} = \mathcal{A}_\kappa^{\otimes M} \circ \mathcal{N}_\kappa^{\otimes M}. \quad (46)$$

Proof. It is sufficient to prove that $\mathcal{M}_\kappa^{\otimes M}(\hat{\rho})$ and $\mathcal{A}_\kappa^{\otimes M}(\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}))$ have the same Husimi Q representation for any quantum state $\hat{\rho}$. Let us fix $\mathbf{w} \in \mathbb{C}^M$. On one hand we have from (39) and (7)

$$\begin{aligned} \langle \mathbf{w} | \mathcal{M}_\kappa^{\otimes M}(\hat{\rho}) | \mathbf{w} \rangle &= \int_{\mathbb{C}^M} \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle |\langle \sqrt{\kappa} \mathbf{z} | \mathbf{w} \rangle|^2 \frac{d^{2M}z}{\pi^M} \\ &= \int_{\mathbb{C}^M} e^{-|\mathbf{w} - \sqrt{\kappa} \mathbf{z}|^2} \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle \frac{d^{2M}z}{\pi^M}. \end{aligned} \quad (47)$$

On the other hand we have from Lemma 4 and Eqs. (45) and (44)

$$\begin{aligned} \langle \mathbf{w} | \mathcal{A}_\kappa^{\otimes M}(\mathcal{N}_\kappa^{\otimes M}(\hat{\rho})) | \mathbf{w} \rangle &= \text{Tr} [\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) \mathcal{A}_\kappa^{\otimes M \dagger}(|\mathbf{w}\rangle\langle \mathbf{w}|)] \\ &= \frac{\langle \mathbf{w}/\sqrt{\kappa} | \mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) | \mathbf{w}/\sqrt{\kappa} \rangle}{\kappa^M} \\ &= \int_{\mathbb{C}^M} e^{-\kappa|\mathbf{z}|^2} \langle \mathbf{w}/\sqrt{\kappa} | \hat{D}(\mathbf{z}) \hat{\rho} \hat{D}(\mathbf{z})^\dagger | \mathbf{w}/\sqrt{\kappa} \rangle \frac{d^{2M}z}{\pi^M} \\ &= \int_{\mathbb{C}^M} e^{-\kappa|\mathbf{z}|^2} \langle \mathbf{w}/\sqrt{\kappa} - \mathbf{z} | \hat{\rho} | \mathbf{w}/\sqrt{\kappa} - \mathbf{z} \rangle \frac{d^{2M}z}{\pi^M} \\ &= \int_{\mathbb{C}^M} e^{-|\mathbf{w} - \sqrt{\kappa} \mathbf{z}|^2} \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle \frac{d^{2M}z}{\pi^M}. \end{aligned} \quad (48)$$

Lemma 2. For any quantum state $\hat{\rho}$

$$\lim_{\kappa \rightarrow \infty} \|\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) - \hat{\rho}\|_1 = 0. \quad (49)$$

Proof. We have from (45)

$$\begin{aligned} \|\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) - \hat{\rho}\|_1 &= \left\| \int_{\mathbb{C}^M} e^{-|z|^2} \left(\hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right) \hat{\rho} \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right)^\dagger - \hat{\rho} \right) \frac{d^{2M}z}{\pi^M} \right\|_1 \\ &\leq \int_{\mathbb{C}^M} e^{-|z|^2} \left\| \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right) \hat{\rho} \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right)^\dagger - \hat{\rho} \right\|_1 \frac{d^{2M}z}{\pi^M}. \end{aligned} \quad (50)$$

The integrands are dominated by

$$\int_{\mathbb{C}^M} 2e^{-|z|^2} \frac{d^{2M}z}{\pi^M} = 2, \quad (51)$$

and from Lemma 6

$$\lim_{\kappa \rightarrow \infty} \left\| \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right) \hat{\rho} \hat{D}\left(\frac{\mathbf{z}}{\sqrt{\kappa}}\right)^\dagger - \hat{\rho} \right\|_1 = 0. \quad (52)$$

The claim then follows from the dominated convergence theorem.

We have from Klein's inequality applied to $\hat{A} = \kappa^M \mathcal{A}_\kappa^{\otimes M}(\mathcal{N}_\kappa(\hat{\rho}))$ and $\hat{B} = \kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho})$ and (42)

$$\begin{aligned} \int_{\mathbb{C}^M} f(\langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle) \frac{d^{2M}z}{\pi^M} &\geq \limsup_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa^M \mathcal{A}_\kappa^{\otimes M}(\mathcal{N}_\kappa^{\otimes M}(\hat{\rho})))}{\kappa^M} \\ &\geq \limsup_{\kappa \rightarrow \infty} \left(\frac{\text{Tr } f(\kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}))}{\kappa^M} - \|\mathcal{A}_\kappa^{\otimes M}(\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) - \hat{\rho})\|_1 \|f'\|_\infty \right). \end{aligned} \quad (53)$$

From the contractivity of the trace norm under quantum channels and from Lemma 2 we get

$$\limsup_{\kappa \rightarrow \infty} \|\mathcal{A}_\kappa^{\otimes M}(\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) - \hat{\rho})\|_1 \leq \limsup_{\kappa \rightarrow \infty} \|\mathcal{N}_\kappa^{\otimes M}(\hat{\rho}) - \hat{\rho}\|_1 = 0, \quad (54)$$

and the claim (37) follows.

7. Proof of Theorem 5

From Theorem 4

$$\int_{\mathbb{C}} f(\langle z | \hat{\rho} | z \rangle) \frac{d^2z}{\pi} = \lim_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa \mathcal{A}_\kappa(\hat{\rho}))}{\kappa}. \quad (55)$$

From Theorem 1

$$\text{Tr } f(\kappa \mathcal{A}_\kappa(\hat{\rho})) \leq \text{Tr } f(\kappa \mathcal{A}_\kappa(\hat{\rho}^\downarrow)). \quad (56)$$

We then have

$$\int_{\mathbb{C}} f(\langle z | \hat{\rho} | z \rangle) \frac{d^2z}{\pi} \leq \lim_{\kappa \rightarrow \infty} \frac{\text{Tr } f(\kappa \mathcal{A}_\kappa(\hat{\rho}^\downarrow))}{\kappa} = \int_{\mathbb{C}} f(\langle z | \hat{\rho}^\downarrow | z \rangle) \frac{d^2z}{\pi}. \quad (57)$$

8. Proof of Theorem 6

Choosing $f(x) = x^q$ in Theorem 4 we get

$$\|Q(\hat{\rho})\|_q^q = \int_{\mathbb{C}^M} \langle \mathbf{z} | \hat{\rho} | \mathbf{z} \rangle^q \frac{d^{2M}z}{\pi^M} = \lim_{\kappa \rightarrow \infty} \kappa^{M(q-1)} \text{Tr } \mathcal{A}_\kappa^{\otimes M}(\hat{\rho})^q. \quad (58)$$

Conjecture 2 then gives

$$\frac{\|Q(\hat{\rho})\|_q}{\|\hat{\rho}\|_p} = \lim_{\kappa \rightarrow \infty} \kappa^{M \frac{q-1}{q}} \frac{\|\mathcal{A}_\kappa^{\otimes M}(\hat{\rho})\|_q}{\|\hat{\rho}\|_p} \leq \left(\lim_{\kappa \rightarrow \infty} \sup_{0 \leq z < 1} \kappa^{\frac{q-1}{q}} \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} \right)^M. \quad (59)$$

We can compute from (3) for any $0 \leq z < 1$ and any $p \geq 1$

$$\|\hat{\omega}_z\|_p = \frac{1-z}{(1-z^p)^{\frac{1}{p}}}. \quad (60)$$

We have from (15) and (60) for any $0 \leq z < 1$ and any $\kappa \geq 1$

$$\kappa^{\frac{q-1}{q}} \|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q = \frac{1-z}{(\kappa - \kappa(1 - \frac{1-z}{\kappa})^q)^{\frac{1}{q}}}. \quad (61)$$

From (58) with $\hat{\rho} = \hat{\omega}_z$ we get

$$\|Q(\hat{\omega}_z)\|_q = \lim_{\kappa \rightarrow \infty} \kappa^{\frac{q-1}{q}} \|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q = \frac{(1-z)^{\frac{q-1}{q}}}{q^{\frac{1}{q}}} . \quad (62)$$

Let us choose $1 \leq r < q$. For any $0 \leq z < 1$ and any $\kappa \geq \frac{1}{x_r}$, with x_r as in Lemma 7, we have

$$\frac{1-z}{\kappa} \leq \frac{1}{\kappa} \leq x_r . \quad (63)$$

We then have from Lemma 7

$$\left(1 - \frac{1-z}{\kappa}\right)^q \leq 1 - r \frac{1-z}{\kappa} , \quad (64)$$

and from (61) and (62)

$$\kappa^{\frac{q-1}{q}} \|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q \leq \frac{(1-z)^{\frac{q-1}{q}}}{r^{\frac{1}{q}}} = \frac{q^{\frac{1}{q}}}{r^{\frac{1}{q}}} \|Q(\hat{\omega}_z)\|_q . \quad (65)$$

It follows that

$$\lim_{\kappa \rightarrow \infty} \sup_{0 \leq z < 1} \kappa^{\frac{q-1}{q}} \frac{\|\mathcal{A}_\kappa(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} \leq \frac{q^{\frac{1}{q}}}{r^{\frac{1}{q}}} \sup_{0 \leq z < 1} \frac{\|Q(\hat{\omega}_z)\|_q}{\|\hat{\omega}_z\|_p} , \quad (66)$$

and the claim follows taking the limit $r \rightarrow q$.

Let us prove that the supremum in (26) is finite if $1 \leq p \leq q$. We have for any $0 \leq z < 1$

$$\frac{(1-z^p)^{\frac{1}{p}}}{(1-z)^{\frac{1}{q}}} \leq \left(\frac{1-z^p}{1-z}\right)^{\frac{1}{p}} \leq p^{\frac{1}{p}} . \quad (67)$$

On the other hand, if $p > q \geq 1$ we have

$$\lim_{z \rightarrow 1} \frac{(1-z^p)^{\frac{1}{p}}}{(1-z)^{\frac{1}{q}}} = \infty , \quad (68)$$

and the supremum in (26) is infinite.

9. Proof of Theorem 7

The proof of (29) does not require f to be differentiable. We can then choose $f(x) = x \ln x$ and get

$$S(Q(\hat{\rho})) \geq \limsup_{\kappa \rightarrow \infty} (S(\mathcal{A}_\kappa^{\otimes M}(\hat{\rho})) - M \ln \kappa) . \quad (69)$$

We get from Conjecture 3

$$\begin{aligned} S(Q(\hat{\rho})) &\geq M \limsup_{\kappa \rightarrow \infty} \left(g \left(\kappa g^{-1} \left(\frac{S(\hat{\rho})}{M} \right) + \kappa - 1 \right) - \ln \kappa \right) \\ &= M \left(\ln \left(g^{-1} \left(\frac{S(\hat{\rho})}{M} \right) + 1 \right) + 1 \right) , \end{aligned} \quad (70)$$

where we have used that for $x \rightarrow \infty$

$$g(x) = \ln x + 1 + \mathcal{O}\left(\frac{1}{x}\right). \quad (71)$$

We now show that (28) is saturated by the thermal Gaussian states $\hat{\omega}_z^{\otimes M}$, $0 \leq z < 1$. On one hand we have from (5) $S(\hat{\omega}_z^{\otimes M}) = M g(E)$, with E given by (4). On the other hand we have

$$Q(\hat{\omega}_z^{\otimes M})(\mathbf{z}) = \frac{e^{-\frac{|\mathbf{z}|^2}{E+1}}}{(E+1)^M}, \quad (72)$$

and

$$S(Q(\hat{\omega}_z^{\otimes M})) = M(\ln(E+1) + 1). \quad (73)$$

10. Conclusions

We have proven that the quantum-classical channel that associates to a quantum state its Husimi Q representation is asymptotically equivalent to the Gaussian quantum-limited amplifier with infinite amplification parameter (Theorem 4). This equivalence has permitted us to determine the minimum Wehrl entropy among all the quantum states with a given von Neumann entropy, and prove that it is achieved by a thermal Gaussian state (Theorem 7). This result determines the relation between the von Neumann and the Wehrl entropies. The same equivalence has also permitted us to determine the $p \rightarrow q$ norms of the aforementioned quantum-classical channel, and prove that they are achieved by thermal Gaussian states (Theorem 6).

The Husimi Q representation of a quantum state coincides with the probability distribution of the outcome of a heterodyne measurement performed on the state. Then, our results can find applications in quantum cryptography for the quantum key distribution schemes based on the heterodyne measurement [6, 7].

The proofs of Theorems 6 and 7 assume Conjectures 2 and 3, respectively, that have been proven for one mode only. The multimode version of Theorems 7 and 6 is implied by the multimode version of Conjectures 2 and 3, whose proof will be the subject of future work.

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A. The Gaussian quantum-limited attenuator

The M -mode Gaussian quantum-limited attenuator $\mathcal{E}_\lambda^{\otimes M}$ with attenuation parameter $0 \leq \lambda \leq 1$ is the quantum channel that mixes through a beamsplitter with transmissivity λ the input state $\hat{\rho}$ and the vacuum state of a M -mode ancillary Gaussian system B with ladder operators $\hat{b}_1, \dots, \hat{b}_M$:

$$\mathcal{E}_\lambda^{\otimes M}(\hat{\rho}) = \text{Tr}_B \left[\hat{U}_\lambda (\hat{\rho} \otimes |\mathbf{0}\rangle\langle\mathbf{0}|) \hat{U}_\lambda^\dagger \right]. \quad (74)$$

The beamsplitter is implemented by the two-mode mixing unitary operator

$$\hat{U}_\lambda = \exp \left(\arccos \sqrt{\lambda} \sum_{i=1}^M \left(\hat{a}_i^\dagger \hat{b}_i - \hat{a}_i \hat{b}_i^\dagger \right) \right), \quad (75)$$

and it acts on the ladder operators as

$$\hat{U}_\lambda^\dagger \hat{a}_i \hat{U}_\lambda = \sqrt{\lambda} \hat{a}_i + \sqrt{1-\lambda} \hat{b}_i, \quad (76)$$

$$\hat{U}_\lambda^\dagger \hat{b}_i \hat{U}_\lambda = -\sqrt{1-\lambda} \hat{a}_i + \sqrt{\lambda} \hat{b}_i, \quad i = 1, \dots, M. \quad (77)$$

Lemma 3 ([22]). *The Gaussian quantum-limited attenuator preserves the set of coherent states, i.e. for any $0 \leq \lambda \leq 1$ and any $\mathbf{z} \in \mathbb{C}^M$*

$$\mathcal{E}_\lambda^{\otimes M}(|\mathbf{z}\rangle\langle\mathbf{z}|) = |\sqrt{\lambda}\mathbf{z}\rangle\langle\sqrt{\lambda}\mathbf{z}|. \quad (78)$$

The relation with the amplifier is given by

Theorem 10 ([34], **Theorem 9**). *The Gaussian quantum-limited attenuator and amplifier are mutually dual, i.e. for any $\kappa \geq 1$*

$$\kappa^M \mathcal{A}_\kappa^{\otimes M \dagger} = \mathcal{E}_{\frac{1}{\kappa}}^{\otimes M}. \quad (79)$$

Lemma 4. *For any $\kappa \geq 1$ and any $\mathbf{z} \in \mathbb{C}^M$*

$$\kappa^M \mathcal{A}_\kappa^{\otimes M \dagger}(|\mathbf{z}\rangle\langle\mathbf{z}|) = |\mathbf{z}/\sqrt{\kappa}\rangle\langle\mathbf{z}/\sqrt{\kappa}|. \quad (80)$$

Proof. Follows from Theorem 10 and Lemma 3.

Lemma 5. *For any quantum state $\hat{\rho}$ and any $\kappa \geq 1$*

$$0 \leq \kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}) \leq \hat{\mathbb{I}}. \quad (81)$$

Proof. We have from Theorem 10

$$0 \leq \kappa^M \mathcal{A}_\kappa^{\otimes M}(\hat{\rho}) = \mathcal{E}_{\frac{1}{\kappa}}^{\otimes M \dagger}(\hat{\rho}) \leq \mathcal{E}_{\frac{1}{\kappa}}^{\otimes M \dagger}(\hat{\mathbb{I}}) = \hat{\mathbb{I}}, \quad (82)$$

where we have used that, since the Gaussian quantum-limited attenuator is trace-preserving, its dual is unital.

B. Auxiliary theorems and lemmas

Theorem 11 (Klein's inequality). *Let $f \in C^1([0, 1])$ be a real convex function with $f(0) = 0$. Then, for any two trace-class operators $0 \leq \hat{A}, \hat{B} \leq \hat{\mathbb{I}}$*

$$\mathrm{Tr} f(\hat{B}) \leq \mathrm{Tr} f(\hat{A}) + \left\| \hat{B} - \hat{A} \right\|_1 \|f'\|_\infty. \quad (83)$$

Proof. Let us diagonalize \hat{A} and \hat{B} :

$$\begin{aligned}\hat{A} &= \sum_{m \in \mathbb{N}} a_m |\phi_m\rangle \langle \phi_m|, \quad 0 \leq a_m \leq 1, \quad \langle \phi_m | \phi_{m'} \rangle = \delta_{mm'}, \quad \sum_{m \in \mathbb{N}} |\phi_m\rangle \langle \phi_m| = \hat{\mathbb{I}} \\ \hat{B} &= \sum_{n \in \mathbb{N}} b_n |\psi_n\rangle \langle \psi_n|, \quad 0 \leq b_n \leq 1, \quad \langle \psi_n | \psi_{n'} \rangle = \delta_{nn'}, \quad \sum_{n \in \mathbb{N}} |\psi_n\rangle \langle \psi_n| = \hat{\mathbb{I}}.\end{aligned}\tag{84}$$

Since f is convex, for any $0 \leq a, b \leq 1$

$$f(b) \leq f(a) + (b - a) f'(b). \tag{85}$$

We then have

$$\begin{aligned}\mathrm{Tr} f(\hat{B}) &= \sum_{n \in \mathbb{N}} f(b_n) = \sum_{m, n \in \mathbb{N}} |\langle \phi_m | \psi_n \rangle|^2 f(b_n) \\ &\leq \sum_{m, n \in \mathbb{N}} |\langle \phi_m | \psi_n \rangle|^2 (f(a_m) + (b_n - a_m) f'(b_n)) \\ &= \sum_{m \in \mathbb{N}} f(a_m) + \sum_{n \in \mathbb{N}} b_n f'(b_n) - \sum_{m, n \in \mathbb{N}} |\langle \phi_m | \psi_n \rangle|^2 a_m f'(b_n) \\ &= \mathrm{Tr} \left[f(\hat{A}) + (\hat{B} - \hat{A}) f'(\hat{B}) \right] \\ &\leq \mathrm{Tr} f(\hat{A}) + \|\hat{B} - \hat{A}\|_1 \|f'\|_\infty.\end{aligned}\tag{86}$$

Lemma 6. For any quantum state $\hat{\rho}$

$$\lim_{\mathbf{z} \rightarrow \mathbf{0}} \left\| \hat{D}(\mathbf{z}) \hat{\rho} \hat{D}(\mathbf{z})^\dagger - \hat{\rho} \right\|_1 = 0. \tag{87}$$

Proof. Let us diagonalize $\hat{\rho}$:

$$\hat{\rho} = \sum_{n \in \mathbb{N}} p_n |\psi_n\rangle \langle \psi_n|, \quad p_n \geq 0, \quad \sum_{n \in \mathbb{N}} p_n = 1, \quad \langle \psi_m | \psi_n \rangle = \delta_{mn}. \tag{88}$$

We have for any $\mathbf{z} \in \mathbb{C}^M$

$$\begin{aligned}\left\| \hat{D}(\mathbf{z}) \hat{\rho} \hat{D}(\mathbf{z})^\dagger - \hat{\rho} \right\|_1 &= \left\| \sum_{n \in \mathbb{N}} p_n \left(\hat{D}(\mathbf{z}) |\psi_n\rangle \langle \psi_n| \hat{D}(\mathbf{z})^\dagger - |\psi_n\rangle \langle \psi_n| \right) \right\|_1 \\ &\leq \sum_{n \in \mathbb{N}} p_n \left\| \hat{D}(\mathbf{z}) |\psi_n\rangle \langle \psi_n| \hat{D}(\mathbf{z})^\dagger - |\psi_n\rangle \langle \psi_n| \right\|_1 \\ &= \sum_{n \in \mathbb{N}} 2p_n \sqrt{1 - \left| \langle \psi_n | \hat{D}(\mathbf{z}) | \psi_n \rangle \right|^2}.\end{aligned}\tag{89}$$

The sums are dominated by

$$\sum_{n \in \mathbb{N}} 2p_n = 2. \tag{90}$$

Since $\hat{D}(\mathbf{z})$ is strongly continuous in \mathbf{z} [35], it is also weakly continuous, and we have for any $n \in \mathbb{N}$

$$\lim_{\mathbf{z} \rightarrow \mathbf{0}} \langle \psi_n | \hat{D}(\mathbf{z}) | \psi_n \rangle = 1 . \quad (91)$$

The claim then follows from the dominated convergence theorem.

Lemma 7. For any $1 \leq r < q$ and any $0 \leq x \leq x_r$, with

$$x_r := 1 - \left(\frac{r}{q} \right)^{\frac{1}{q-1}} > 0 , \quad (92)$$

we have

$$(1 - x)^q \leq 1 - r x . \quad (93)$$

Proof. Let us define

$$\phi(x) := 1 - r x - (1 - x)^q . \quad (94)$$

We have $\phi(0) = 0$, and

$$\phi'(x) = q(1 - x)^{q-1} - r . \quad (95)$$

The claim follows since $\phi'(x) \geq 0$ for any $0 \leq x \leq x_r$.

$$\|\hat{\omega}_z\|_p = \frac{1 - z}{(1 - z^p)^{\frac{1}{p}}} \quad (96)$$

$$w = 1 - \frac{1 - z}{\kappa} , \quad 1 - w = \frac{1 - z}{\kappa} \quad (97)$$

References

1. K. Husimi, “Some formal properties of the density matrix,” *Nippon Sugaku-Buturigakkwai Kizi Dai 3 Ki*, vol. 22, no. 4, pp. 264–314, 1940.
2. U. Leonhardt, *Measuring the quantum state of light*. Cambridge university press, 1997, vol. 22.
3. S. Barnett and P. Radmore, *Methods in Theoretical Quantum Optics*, ser. Oxford Series in Optical and Imaging Sciences. Clarendon Press, 2002.
4. W. Schleich, *Quantum Optics in Phase Space*. Wiley, 2015.
5. H. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations*, ser. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2013.
6. C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, “Gaussian quantum information,” *Reviews of Modern Physics*, vol. 84, no. 2, p. 621, 2012.
7. C. Weedbrook, A. M. Lance, W. P. Bowen, T. Symul, T. C. Ralph, and P. K. Lam, “Quantum cryptography without switching,” *Physical review letters*, vol. 93, no. 17, p. 170504, 2004.
8. D. J. Callaway, “On the remarkable structure of the superconducting intermediate state,” *Nuclear Physics B*, vol. 344, no. 3, pp. 627–645, 1990.
9. A. Wehrl, “On the relation between classical and quantum-mechanical entropy,” *Reports on Mathematical Physics*, vol. 16, no. 3, pp. 353–358, 1979.
10. —, “General properties of entropy,” *Reviews of Modern Physics*, vol. 50, no. 2, p. 221, 1978.
11. T. Cover and J. Thomas, *Elements of Information Theory*, ser. A Wiley-Interscience publication. Wiley, 2006.
12. E. H. Lieb, “Proof of an entropy conjecture of wehrl,” *Communications in Mathematical Physics*, vol. 62, no. 1, pp. 35–41, 1978.

13. E. A. Carlen, “Some integral identities and inequalities for entire functions and their application to the coherent state transform,” *Journal of functional analysis*, vol. 97, no. 1, pp. 231–249, 1991.
14. E. Schrödinger, “Der stetige übergang von der mikro-zur makromechanik,” *Naturwissenschaften*, vol. 14, no. 28, pp. 664–666, 1926.
15. V. Bargmann, “On a hilbert space of analytic functions and an associated integral transform part i,” *Communications on pure and applied mathematics*, vol. 14, no. 3, pp. 187–214, 1961.
16. J. R. Klauder, “The action option and a feynman quantization of spinor fields in terms of ordinary c-numbers,” *Annals of Physics*, vol. 11, no. 2, pp. 123–168, 1960.
17. R. J. Glauber, “Coherent and incoherent states of the radiation field,” *Physical Review*, vol. 131, no. 6, p. 2766, 1963.
18. J. Klauder and E. Sudarshan, *Fundamentals of Quantum Optics*, ser. Dover books on physics. Dover Publications, 2006.
19. E. H. Lieb and J. P. Solovej, “Proof of the wehrl-type entropy conjecture for symmetric $\{SU(N)\}$ coherent states,” *Communications in Mathematical Physics*, pp. 1–12, 2015.
20. —, “Proof of an entropy conjecture for bloch coherent spin states and its generalizations,” *Acta Mathematica*, vol. 212, no. 2, pp. 379–398, 2014.
21. V. Giovannetti, A. S. Holevo, and A. Mari, “Majorization and additivity for multimode bosonic gaussian channels,” *Theoretical and Mathematical Physics*, vol. 182, no. 2, pp. 284–293, 2015.
22. A. S. Holevo, “Gaussian optimizers and the additivity problem in quantum information theory,” *Uspekhi Matematicheskikh Nauk*, vol. 70, no. 2, pp. 141–180, 2015.
23. G. De Palma, D. Trevisan, and V. Giovannetti, “Passive states optimize the output of bosonic gaussian quantum channels,” *IEEE Transactions on Information Theory*, vol. 62, no. 5, pp. 2895–2906, May 2016.
24. —, “Gaussian states minimize the output entropy of the one-mode quantum attenuator,” *IEEE Transactions on Information Theory*, vol. 63, no. 1, pp. 728–737, 2017.
25. —, “One-mode quantum-limited gaussian channels have gaussian maximizers,” *arXiv preprint arXiv:1610.09967*, 2016.
26. —, “Gaussian states minimize the output entropy of one-mode quantum gaussian channels,” *arXiv preprint arXiv:1610.09970*, 2016.
27. G. De Palma, A. Mari, S. Lloyd, and V. Giovannetti, “Passive states as optimal inputs for single-jump lossy quantum channels,” *Physical Review A*, vol. 93, no. 6, p. 062328, 2016.
28. R. Schatten, *Norm Ideals of Completely Continuous Operators*, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1960.
29. A. S. Holevo, “Multiplicativity of p-norms of completely positive maps and the additivity problem in quantum information theory,” *Russian Mathematical Surveys*, vol. 61, no. 2, p. 301, 2006.
30. G. De Palma, A. Mari, and V. Giovannetti, “A generalization of the entropy power inequality to bosonic quantum systems,” *Nature Photonics*, vol. 8, no. 12, pp. 958–964, 2014.
31. V. Giovannetti, A. Holevo, and R. García-Patrón, “A solution of gaussian optimizer conjecture for quantum channels,” *Communications in Mathematical Physics*, vol. 334, no. 3, pp. 1553–1571, 2015.
32. G. De Palma, A. Mari, S. Lloyd, and V. Giovannetti, “Multimode quantum entropy power inequality,” *Physical Review A*, vol. 91, no. 3, p. 032320, 2015.
33. F. A. Berezin, “Covariant and contravariant symbols of operators,” *Izvestiya: Mathematics*, vol. 6, no. 5, pp. 1117–1151, 1972.
34. J. S. Ivan, K. K. Sabapathy, and R. Simon, “Operator-sum representation for bosonic gaussian channels,” *Physical Review A*, vol. 84, no. 4, p. 042311, 2011.
35. A. S. Holevo, *Quantum Systems, Channels, Information: A Mathematical Introduction*, ser. De Gruyter Studies in Mathematical Physics. De Gruyter, 2013.